§2. Modules

§2.1 modules and module homomorphisms

A = ring

A-module = abelian group $(M,+)$ with A-linear action. i.e.

$$\mu : A \times M \longrightarrow M$$
$$(a, m) \longmapsto \mu(a,m) =: am$$

satisfying

$$a(x+y) = ax + ay$$
$$(a+b)x = ax + bx$$
$$(ab)x = a(bx)$$
$$1 \cdot x = x$$

$$\Leftrightarrow \quad A \longrightarrow \mathrm{End}(M)$$

ring homomorphism

可能非交换

例: 1). $\mathfrak{a} \lhd A$, (A is an A-module)

2). k-module $\Leftrightarrow$ k-vector space.

3). $\mathbb{Z}$-module $\Leftrightarrow$ abelian group

4). $k[x]$-module $\Leftrightarrow$ k-vector space with a linear transformation

5). G = f.g.p.

$k[G]$-module $\Leftrightarrow$ k-rep. of G.

可能非交换.

<u>A-module homomorphism (A-linear)</u>. A mapping $f: M \to N$ between two A-modules satisfying:

$$f(x+y) = f(x) + f(y) \quad \forall x, y \in M. \qquad \text{(与群结构相容)}$$

$$f(ax) = a \cdot f(x) \quad \forall a \in A, \forall x \in M. \qquad \text{(与模结构相容)}$$

eg. $A = $ field $\Rightarrow$ A-module homomorphism $\Leftrightarrow$ A-linear transformation.

$$\mathrm{Hom}_A(M,N) = \{ f: M \to N \mid f \text{ is an A-module hom.} \}$$

(or $\mathrm{Hom}(M,N)$)

A-module str. on $\mathrm{Hom}_A(M,N)$

$$(f+g)(x) := f(x) + g(x) \quad \forall x$$

$$(af)(x) := a \cdot f(x) \quad \forall x$$

$$\begin{tikzcd} M' \arrow[r, "u"] \arrow[dr, "f \circ u"'] & M \arrow[d, "f"] \arrow[dl, phantom, "\leftarrow"] \\ & N \end{tikzcd} \Rightarrow \bar{u} : \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N)$$
$$f \mapsto f \circ u$$

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "v \circ f"] \arrow[d, phantom, shift left=3ex, "\rightarrow"] & \\ N \arrow[r, "v"] & N'' \end{tikzcd} \Rightarrow \bar{v} : \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N'')$$
$$f \mapsto v \circ f$$

Fact: $\bar{u}$ & $\bar{v}$ are A-module hom.

pf: $\bar{u}(f+g)(m') = (f+g)(u(m')) = f \circ u(m') + g \circ u(m')$
$= \bar{u}(f)(m') + \bar{u}(g)(m') = (\bar{u}(f) + \bar{u}(g))(m') \quad \forall m'$

$\bar{u}(af)(m') = (af)(u(m')) = a \cdot f(u(m')) = a(\bar{u}(f))(m')$
$= (a \cdot \bar{u}(f))(m') \quad \forall m'$

Fact: $M = A$-module $\Rightarrow \mathrm{Hom}(A,M) \xrightarrow{\sim} M \quad f \mapsto f(1)$

pf: $f : A \to M$ is uniquely determined by $f(1)$.

§2.1. Submodules and quotient modules.

submodule = subgroup closed under multiplication by elements of A.

Let $M' \subseteq M$ be a submod. i.e. $(M', +) \subseteq (M, +)$, $AM' \subseteq M'$.

the quotient of M by M'.

$$M/M' = \{x + M' \mid x \in M\}$$

$$a(x + M') := ax + M' \qquad (A\text{-module str.})$$

Fact : 1) $M \to M/M'$ is an A-module homomorphism.

2) $\{M'' \mid M' \subseteq M'' \subseteq M \text{ submodules}\} \xleftrightarrow{1:1} \{N \subseteq M/M' \text{ submod.}\}$

order-preserving.

$f : M \to N$

kernel: $\operatorname{Ker} f := f^{-1}(0) = \{x \in M \mid f(x) = 0\}$

image: $\operatorname{Im} f := f(M)$

cokernel: $\operatorname{Coker} f := N / \operatorname{Im} f$

Fact : 1) $\operatorname{Ker} f \subseteq M$ submodule

2) $M/\operatorname{Ker} f \cong \operatorname{Im} f \subseteq N$ submodule

3). $\operatorname{Coker} f \twoheadleftarrow N$ quotient module

Fact: $M' \subseteq M$ submod. $M' \subseteq \ker f$

$$\Rightarrow \exists!\ \bar{f} : M/M' \to N \quad \text{s.t.} \quad \begin{tikzcd} M \arrow[r, two heads, "f"] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowright"] & N \\ M/M' \arrow[ur, dashed, two heads, "\exists!\,\bar{f}"'] & \end{tikzcd}$$

Pf 1° $\bar{f}(x+M') := f(x)$ (well-defined)

$$x+M' = y+M' \Rightarrow f(x) = f(y)$$

2° $\bar{f}$ is an A-module hom. $\square$

§2.3 operations on submodules

$M_i \subseteq M$ submodules $i \in I$. 可能无限

Sum: $\sum_{i\in I} M_i := \left\{ \sum_{i\in I} x_i \;\middle|\; \begin{array}{l} x_i \in M_i \\ \text{almost all } x_i \text{ are zero} \end{array} \right\}$

intersection: $\bigcap_{i\in I} M_i := \{ x \in M \mid x \in M_i, \forall i \}$

Fact: $\sum M_i$, $\bigcap M_i$ are submodules.

Prop 2.1 i) $N \subseteq M \subseteq L \Rightarrow (L/N)/(M/N) \cong L/M$ (A-mod)

ii) $M_1, M_2 \subseteq M \Rightarrow (M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$ (A-mod)

Pf: $\left. \begin{array}{l} L/N \xrightarrow{\pi} L/M \\ \ker \pi = M/N \end{array} \right\} \Rightarrow$ i)

$\ker\left(M_2 \to M_1 + M_2 \to (M_1+M_2)/M_1 \right) = M_1 \cap M_2 \Rightarrow$ ii)

Product: Let $\mathfrak{a} \trianglelefteq A$, $M = A$-module

$\mathfrak{a}M := \left\{ \sum_{i=1}^{n} a_i m_i \;\middle|\; n \in \mathbb{N}, a_i \in \mathfrak{a}, m_i \in M \right\}$

is a submodule of M

$\underline{(N:P)}$: $N, P \subseteq M$ two submod.

$$(N:P) := \{ a \in A \mid aP \subseteq N \}$$

is an ideal of A.

<u>annihilator</u>:

$$\mathrm{Ann}(M) := (0:M) \trianglelefteq A$$

Fact: $\mathfrak{a} \subseteq \mathrm{Ann}(M) \Rightarrow$ regard M as an $A/\mathfrak{a}$-module.

An A-module is <u>faithful</u> if $\mathrm{Ann}(M)=0$. i.e.

$$aM=0 \overset{\forall a \in A}{\Longleftrightarrow} a=0.$$

Fact: M is faithful as an $A/\mathrm{Ann}(M)$-module.

<u>Lemma</u>: i) $\mathrm{Ann}(M+N) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$

ii) $(N:P) = \mathrm{Ann}((N+P)/N)$

$(x) := Ax \subseteq M \quad \forall x \in M.$

$M = \sum_i Ax_i \Rightarrow \{x_i\}_{i \in I}$ a set of generators of M

$M =$ f.g. if it has a finite set of generators.

§2.4. Direct sum and product

Let M, N be two R-modules.

$$M \oplus N := \{(m,n) \mid m \in M,\ n \in N\}$$

$$(x_1, y_1) + (x_2, y_2) = \cdots$$

$$a(x,y) := (ax, ay) \qquad \forall a$$

$\Rightarrow$ $M \oplus N$ forms an R-module

Similarly. $\forall\ (x_i)_{i \in I} \in \bigoplus_{i \in I} M_i$ or $\forall\ (x_i)_{i\in I} \in \prod_{i \in I} M_i$

$$a \cdot \left((x_i)_{i \in I}\right) := (a x_i)_{i \in I}$$

$\Rightarrow$ $\bigoplus_{i \in I} M_i$ and $\prod_{i\in I} M_i$ form R-modules

Fact: $\# I < \infty \Rightarrow \bigoplus_{i \in I} M_i = \prod_{i \in I} M_i$

Fact: $A = \prod_{i=1}^n A_i \overset{①}{\Leftrightarrow} A = \bigoplus_{i=1}^n \mathfrak{a}_i \overset{②}{\Leftrightarrow} 1 = \sum_{i=1}^n e_i,\ e_i^2 = e_i.$

as ring   as ideals

① $\Rightarrow$) $\mathfrak{a}_i = 0 \times \cdots \times A_i \times \cdots \times 0 \lhd A$

① $\Leftarrow$) $A_i := A / \bigoplus_{j \neq i} \mathfrak{a}_j$

② $\Rightarrow$) $\quad e_i e_j \in I_i \cap I_j = 0 \quad \forall i \neq j.$

② $\Leftarrow$) $\quad \begin{cases} 1 = \sum e_i \\ e_i^2 = e_i \end{cases} \Rightarrow e_i e_j = 1 \Rightarrow (I_1 \oplus \cdots \oplus I_i) \cap I_{i+1} = 0.$

§2.5. finitely generated modules

An A-module M is called free, if

$$M \cong \bigoplus_{i\in I} A =: A^{(I)}$$

notation: $A^{(I)}$ or A^n if $|I|=n$

Prop 2.3: $M =$ f.g. A-module $\Longleftrightarrow$ $M \cong$ a quotient of some A^n.

Pf: $\Leftarrow$) $A^n =$ f.g. $\Rightarrow$ quot. of $A^n =$ f.g $\Rightarrow$ $M =$ f.g.

$\Rightarrow$) Let $x_1 \cdots x_n$ be a system of generators.

$$\phi : A^n \twoheadrightarrow M$$

$$(a_1, \cdots, a_n) \mapsto a_1x_1 + \cdots + a_nx_n$$

$\Rightarrow$ $M \cong A^n / \ker \phi$.

Prop 2.4. $M =$ f.g. A-module.

$\mathfrak{a} \trianglelefteq A$.

$\phi \in \mathrm{End}_A(M)$ s.t. $\phi(M) \subseteq \mathfrak{a}M$

Then $\exists\, a_1, a_2, \cdots, a_n \in \mathfrak{a}$ s.t.

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0 \in \mathrm{End}_A(M).$$

Pf: $M = \sum_{i=1}^{n} A x_i$

$$\Rightarrow \phi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad B \in \mathfrak{a}^{n\times n}$$

$$\Rightarrow (\phi \cdot I_n - B) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \qquad \det(\phi I_n - B)\cdot I_n = (\phi I_n - B)^* \cdot (\phi I_n - B)$$

$$\Rightarrow \det(\phi I_n - B) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$$\Rightarrow \phi^n + a_1 \phi^{n-1} + \cdots + a_n = 0 \quad \text{in } \mathrm{End}_A(M).$$

with $a_i \in \mathfrak{a}$.

Cor 2.5: M = f.g. A-mod. $\mathfrak{a} \lhd A$

$\mathfrak{a} M = M \Rightarrow \exists\, x \equiv 1 \ (\mathfrak{a})$ s.t. $xM = 0$.

Pf: $\phi = \mathrm{id} : M \to \mathfrak{a} M$, $x := 1 + a_1 + \cdots + a_n \in R$

$\Rightarrow xM = 0$. $\square$

Cor 2.6 (Nakayama's lemma). M = f.g. $\mathfrak{a} \subseteq \mathrm{Rad}(A)$.

$\mathfrak{a} M = M \Rightarrow M = 0$.

Pf ①: $\mathfrak{a} \subseteq \mathrm{Rad}(A) \Rightarrow x = 1 + a_1 + \cdots + a_n \in R^\times$.

$$xM = 0 \Rightarrow M = x^{-1} \cdot xM = 0. \qquad \square$$

Pf ②: $M \neq 0 \Rightarrow$ minimal generators $u_1, u_2, \ldots, u_n$

$$u_n \in \mathfrak{a}M \Rightarrow u_n = a_1 x_1 + \cdots + a_n x_n \quad a_i \in \mathrm{Rad}(A)$$

$$\Rightarrow (1 - a_n) u_n \in \sum_{i=1}^{n-1} A u_i$$

$$\xRightarrow{1 - a_n \in A^\times} u_n \in \sum_{i=1}^{n-1} A u_i \quad ↯$$

Cor 2.7: $M =$ f.g. $N \subseteq M$. $\mathfrak{a} \leq \mathrm{Rad}(A)$.

$$M = \mathfrak{a}M + N \Rightarrow M = N$$

$$\Downarrow \qquad\qquad \Uparrow$$

Pf: $$M/N = \mathfrak{a}(M/N) \Rightarrow M/N = 0$$

$(A, m, k) =$ local ring.

$M =$ f.g. A-mod. $\Rightarrow M/mM =$ f.dim. k-vs.

Let $x_1, \ldots, x_n \in M$

Prop 2.8: $M/mM = \sum_{i=1}^{n} k \cdot \bar{x}_i \Rightarrow M = \sum_{i=1}^{n} A \cdot x_i$

Pf: $N := \sum_{i=1}^{n} A x_i$

$$M/mM = \sum_{i=1}^{n} k\overline{x_i} \Rightarrow M = mM + N \qquad \square$$

§2.6 Exact sequences

$$\cdots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots$$

is called <u>exact</u>, if

$$\ker f_{i+1} = \mathrm{Im} f_i \qquad \forall i$$

e.g. (1). $0 \to M' \xrightarrow{f} M$ exact $\Leftrightarrow$ $f = \mathrm{inj}$.

(2). $M \xrightarrow{g} M'' \to 0$ exact $\Leftrightarrow$ $g = \mathrm{surj}$.

(3). $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ exact $\Leftrightarrow$ $\begin{cases} f = \mathrm{inj} \\ g = \mathrm{surj} \end{cases}$

$M/M' \xrightarrow{\sim} M''$

short exact sequence.

Any long exact sequence split into short ones in the following way.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{i-1} \arrow[rr, "f_i"] \arrow[dr] & & M_i \arrow[rr, "f_{i+1}"] \arrow[dr] & & M_{i+1} \arrow[r] & \cdots \\
& 0 \arrow[r] & \mathrm{Im} f_i = \mathrm{Ker} f_{i+1} \arrow[ur] & & \mathrm{Im} f_{i+1} = \mathrm{Ker} f_{i+2} \arrow[ur] \arrow[r] & 0
\end{tikzcd}$$

Prop 2.9: i) $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ (*) is exact, if and only if $\forall N$, $0 \to \mathrm{Hom}(M'', N) \xrightarrow{\bar{v}} \mathrm{Hom}(M, N) \xrightarrow{\bar{u}} \mathrm{Hom}(M', N)$ (**) is exact.

Exercise ii) $0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$ is exact if and only if $\forall M$, $0 \to \mathrm{Hom}(M, N') \xrightarrow{\bar{u}} \mathrm{Hom}(M, N) \xrightarrow{\bar{v}} \mathrm{Hom}(M, N'')$ is exact

Pf: i). $\Rightarrow$) 1° $\bar{v} = \mathrm{inj}$ i.e. $\mathrm{Ker}\,\bar{v} = 0$.

$\forall f \in \ker \bar{v}$

$\Rightarrow \quad f \circ v = 0$

$\overset{v = \mathrm{surj}}{\Rightarrow} \quad f = 0$

2° $\mathrm{Ker}\,\bar{u} \subset \mathrm{im}(\bar{v})$

$\forall g \in \mathrm{Ker}\,\bar{u}$

$\Rightarrow \quad g \circ u = 0$

$\Rightarrow \quad \mathrm{im}\,u \subseteq \mathrm{Ker}\,g$

$\Rightarrow \quad \ker v \subseteq \ker g$

$\Rightarrow \exists ! f\ M'' \to N$ s.t.

$g = f \circ v = \bar{v}(f)$

$\Rightarrow g \in \mathrm{im}(\bar{v})$

3° $\mathrm{im}\,\bar{v} \subseteq \mathrm{Ker}\,\bar{u}$ i.e. $\bar{u} \circ \bar{v} = 0$.

$\forall\ f \in \mathrm{Hom}(M'', N)$

$\bar{u} \circ \bar{v}(f) = f \circ v \circ u = 0$

$\Rightarrow \bar{u} \circ \bar{v} = 0$

$1^\circ\ 2^\circ\ 3^\circ \Rightarrow$ (**) exact.

ii) $\Leftarrow$): 1° $v = $ surj.

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[dr, "0"'] & M'' \arrow[d, "\pi"] \\ & \mathrm{Ker}\, v \end{tikzcd}$$

$\Rightarrow \bar{v}(\pi) = \left(M \xrightarrow{v} M'' \xrightarrow{\pi} \mathrm{ker}\, v\right) = 0$

$\xRightarrow{\bar{v} = \mathrm{inj}} \pi = 0 \Rightarrow v = $ surj.

2° $v \circ u = 0$ i.e. $\operatorname{im} u \subset \ker v$

$N := M''$

$$\begin{tikzcd} M' \arrow[r, "u"] \arrow[drr, "0"'] & M \arrow[r, "v"] \arrow[dr] & M'' \arrow[d, "id"] \\ & & M'' \end{tikzcd}$$

$\Rightarrow$ $v \circ u = \bar{u} \circ \bar{v}\,(id_{M''}) = 0$

3° $\ker v \subset \operatorname{im} u$

$$\begin{tikzcd} M' \arrow[r, "u"] \arrow[dr, "0"'] & M \arrow[r, "v"] \arrow[d, "\phi"] & M'' \arrow[dl, dashed, "\exists\psi"] \\ & M/\operatorname{Im} u & \end{tikzcd}$$

$\Rightarrow \ker v \subset \ker \phi$
$\qquad\qquad\quad \parallel$
$\qquad\qquad\quad \operatorname{im} u$

ii) $\Rightarrow$): 1° $\ker \bar{u} = 0$

$\bar{u}(f) = 0 \Rightarrow u \circ f = 0 \xRightarrow{u = inj} f = 0$

2° $\bar{v} \circ \bar{u} = 0$ $(\operatorname{im} \bar{u} \subset \ker \bar{v})$

$\bar{v} \circ \bar{u}\,(f) = v \circ u \circ f = 0$

3° $\ker \bar{v} \subset \operatorname{Im} \bar{u}$

$\forall f \in \ker \bar{v}$

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "0"] & \\ N \arrow[r, "v"] & N'' \end{tikzcd}$$

$\Rightarrow$ $\operatorname{im} f \subset \ker v = \operatorname{im} u = N'$

$\Rightarrow$ $\exists f' = f : M \to N'$ s.t. $\bar{u}(f') = f$

$\Rightarrow$ $f \in \operatorname{Im} \bar{u}$

$\Leftarrow)$: 1° $u = \text{inj}$

$$\begin{tikzcd} \operatorname{Ker} u \arrow[d, "i"'] \arrow[dr, "0"] & \\ N' \arrow[r, "u"] & N \end{tikzcd} \quad \Rightarrow \quad i = 0 \quad \Rightarrow \quad u = \text{inj}$$

2° $v \circ u = 0$ $(\operatorname{Im} u \subset \operatorname{Ker} v)$

$$\begin{tikzcd} N' \arrow[d, "id"'] \arrow[dr] \arrow[drr, "0"] & & \\ N' \arrow[r, "u"'] & N \arrow[r, "v"'] & N'' \end{tikzcd}$$

3° $\operatorname{Im} u \supset \operatorname{Ker} v$

$$\begin{tikzcd} & \operatorname{Ker} v \arrow[dl, dashed, "\exists"'] \arrow[d] \arrow[dr, "0"] & \\ N' \arrow[r] & N \arrow[r, "v"] & N'' \end{tikzcd}$$

Prop 2.10 (snack Lemma).

$$\begin{tikzcd} & 0 \arrow[r] & \operatorname{Ker} f' \arrow[r, "\bar{u}"] \arrow[d, "i'"] & \operatorname{Ker} f \arrow[r, "\bar{v}"] \arrow[d, "i"] & \operatorname{Ker} f'' \arrow[dddll, bend left, "d"] \arrow[d, "i''"] & & \\ 0 \arrow[r] & M' \arrow[r, "u"] & M \arrow[r, "v"] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowleft"] & M'' \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & M'' \arrow[d, "f''"] \arrow[r] & 0 & \text{exact} \\ 0 \arrow[r] & N' \arrow[r, "u'"'] & N' \arrow[r, "v'"'] \arrow[d, "\pi'"] & N \arrow[r] \arrow[d, "\pi"] & N'' \arrow[r] \arrow[d, "\pi''"] & 0 & \text{exact} \\ & & \operatorname{Coker} f' \arrow[r, "\bar{u}'"] & \operatorname{Coker} f \arrow[r, "\bar{v}'"] & \operatorname{Coker} f'' \arrow[r] & 0 & \text{exact} \end{tikzcd}$$

Pf: $\forall\, m'' \in \ker f''$ $\exists\, m \in M$ s.t.

$$\nu(m) = m'' \quad \left(\Rightarrow f(m) \in N'\right)$$

$$d(m'') := \pi'(f(m))$$

If $\tilde{m} \in M$, s.t. $\nu(\tilde{m}) = m''$ $\left(\Rightarrow f(\tilde{m}) \in N'\right)$

$$m - \tilde{m} \in \ker \nu = M'$$

$$\Rightarrow \quad f(m) - f(\tilde{m}) = f'(m - \tilde{m}) \in \ker \pi'$$

$\Rightarrow d$ is well-defined.

exactness : excercise. $\square$

C = a class of A-modules

$$\lambda : C \longrightarrow \mathbb{Z}$$

λ is called <u>additive</u>, if

$$\lambda(M) = \lambda(M') + \lambda(M'')$$

$\forall$ $0 \to M' \to M \to M'' \to 0$ exact.

Prop 2.11 $\quad 0 \to M_0 \to M_1 \to \cdots M_n \to 0$ exact

λ = additive

$$\Rightarrow \sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0$$

Pf:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & M_0 \arrow[r] \arrow[dr] & M_1 \arrow[r] \arrow[dr] & M_2 \arrow[r] \arrow[dr] & \cdots \arrow[r] & M_{n-1} \arrow[r] \arrow[dr] & M_n \arrow[r] \arrow[dr] & 0 \\
0 = N_0 \arrow[ur] & & N_1 \arrow[ul, phantom] \arrow[u] & N_2 \arrow[u] & N_3 \arrow[u] & N_{n-1} \arrow[u] & N_n \arrow[u] & N_{n+1} = 0
\end{tikzcd}$$

$$\Rightarrow \sum_{i=0}^{n} (-1)^i \lambda(M_i) = \sum_{i=0}^{n} (-1)^i \big(\lambda(N_i) - \lambda(N_{i+1})\big)$$

$$= \lambda(N_0) - (-1)^n \lambda(N_{n+1}) = 0$$

§2.7 Tensor product of modules.

$M, N, P =$ A-modules

$f: M \times N \to P$ is called A-bilinear, if

$f(m, an) = a f(m,n) = f(am, n)$ $\forall m \in M, \forall n \in N$ $\forall a \in A$

Prop 2.12: $\forall M, N$, $\exists\ (T, g: M\times N \to T)$ s.t.

$\forall f: M \times N \to P$ bilinear $\exists !\ f': T \to P$ s.t.

$$f' \circ g = f$$

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "\forall f"'] & T =: M \otimes N \arrow[d, "\exists !\, f'"] \\ & P \end{tikzcd}$$

- (T, g) is unique up to an isomorphism.

Pf: Uniqueness:

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "g'"'] & T \arrow[loop above, "id"] \arrow[d, shift right, "\exists!\, j"'] \\ & T \arrow[u, shift right, "\exists!\, j'"'] \arrow[loop below, "id"] \end{tikzcd}$$

existence:

$$C := A^{(M \times N)} := \left\{ \sum_i a_i (m_i, n_i) \;\middle|\; \begin{array}{l} a_i \in A \\ m_i \in M \\ n_i \in N \end{array} \right\}$$

$D :=$ submodule of C generated by

$$(x + x', y) - (x, y) - (x', y)$$

$$(x, y + y') - (x, y) - (x, y')$$

$$(ax, y) - a(x, y)$$

$$(x, ay) - a(x, y)$$

$T := C/D$ & $m \otimes n := \overline{(m, n)} \in T$

$g(m, n) := m \otimes n$

- It's clear that g is A-bilinear

- $\forall f : M \times N \to P \quad \exists \bar{f} : C \to P. \quad m \otimes n \mapsto f(m,n)$

  $\bar{f}$ well-define since $\bar{f}|_D = 0$.

<u>Cor</u>: 2.13. $\quad \sum_i x_i \otimes y_i = 0 \in M \otimes N.$

$\Rightarrow \exists\ M_0 \subset M,\ N_0 \subset N$ f.g. A-modules

s.t.

$$\sum_i x_i \otimes y_i = 0 \in M_0 \otimes N_0$$

pf: $\quad \sum_i x_i \otimes y_i = 0 \Rightarrow \sum_i (x_i, y_i) \in D \Rightarrow \sum_i (x_i, y_i) = \sum_k g_k$

f. sum of generators of D

$M_0 = \langle x_i \mid i \rangle + \langle \alpha \mid \text{first coordinates} \rangle \subseteq M$

$N_0 = \langle y_i \mid i \rangle + \langle \beta \mid \text{second coordinates} \rangle \subseteq N$

$\Rightarrow \sum x_i \otimes y_i = 0 \in M_0 \otimes N_0$

Rmk: i) $M \otimes_A N := T$ (or $M \otimes N$)

$M = \sum_i A m_i \quad N = \sum_j A n_j \Rightarrow M \otimes N = \sum_{i,j} A\, m_i \otimes n_j$

$M, N = \text{f.g.} \Rightarrow M \otimes N = \text{f.g.}$

ii) $M' \hookrightarrow M$ & $N' \hookrightarrow N \not\Rightarrow M' \otimes N' \hookrightarrow M \otimes N$

e.g. $2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$

$\underset{\neq 0}{2 \otimes 1} \longmapsto \underset{= 0}{2 \otimes 1}$

iii) forget explicit construction and remember the universal property.

iv) multilinear mappings & multi-tensor product.

Prop 2.12' $T = M_1 \otimes \cdots \otimes M_r$ (universal property ?)

Prop 2.14 (Canonical isom.) $\exists!$ iso.

i) $M \otimes N \xrightarrow{\sim} N \otimes M \qquad x \otimes y \mapsto y \otimes x$

ii) $(M \otimes N) \otimes P \to M \otimes N \otimes P \to M \otimes (N \otimes P) \qquad (x \otimes y) \otimes z \mapsto x \otimes y \otimes z \mapsto x \otimes (y \otimes z)$

iii) $(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P) \qquad (x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$

iv) $A \otimes M \to M \qquad a \otimes x \mapsto ax$

Exercise 2.15

$$(M_A \otimes_A {}_A N_B) \otimes_B {}_B P \xrightarrow{\sim} M_A \otimes_A ({}_A N_B \otimes_B {}_B P)$$

$f \otimes g$:

Let $f : M \to M'$, $g : N \to N'$.

$$h : M \times N \to M' \otimes N'$$
$$(x, y) \mapsto f(x) \otimes g(y)$$

$\Rightarrow \exists!\ f \otimes g : M \otimes N \to M' \otimes N'$ s.t.

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$$

Fact : $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$

Pf : agree on $x \otimes y \in M \otimes N$

$\Rightarrow$ agree on $M \otimes N$.

§2.8 Restriction and extension of scalars.

$f : A \to B$ ring homomorphism

$N = B$-module

- A-module structure on N : $an := f(a)n \in N$

  restriction of scalars.

  in particular, A-module str. on B !

Prop 2.16 $\left.\begin{array}{l} N = \text{f.g. } B\text{-module} \\ B = \text{f.g as an } A\text{-module} \end{array}\right\} \Rightarrow N = \text{f.g. } A\text{-module.}$

Pf: $\left.\begin{array}{l} N = \sum_{i=1}^{n} B y_i \\ B = \sum_{j=1}^{m} A x_j \end{array}\right\} \Rightarrow N = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A x_j \right) \cdot y_i$

$$= \sum_{i,j} A \cdot (x_j y_i)$$

- $M = A$-module

  $M_B := B \otimes_A M$ (extension of scalars)

  $b(b' \otimes m) := bb' \otimes m$ (B-mod. str.)

Prop 2.17. $M = \text{f.g.}/A \Rightarrow M_B = \text{f.g.}/B$

Pf: $M = \sum_{i=1}^{m} A x_i \Rightarrow M_B = \sum_{i=1}^{m} B \cdot (1 \otimes x_i)$ $\square$

§2.9 Exactness properties of the tensor product

Fact: $\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N,P))$

Pf: • $\forall f : M \times N \to P$ A-bilinear

$$\Rightarrow \quad M \to \mathrm{Hom}(N,P)$$
$$x \mapsto (y \mapsto f(x,y))$$

• $\forall \phi : M \to \mathrm{Hom}(N,P)$ A-hom.

$$\Rightarrow \quad M \times N \to P \qquad (x,y) \mapsto \phi(x)(y)$$

Prop 2.18. (Right exactness) $\underbrace{M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0}_{(*)}$ exact.

$$\Rightarrow \underbrace{M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0}_{(**)} \text{ exact.}$$

Pf: $(*)$ = r. exact $\Rightarrow$ $\mathrm{Hom}(*, \mathrm{Hom}(N,P))$ = l. exact $\forall P$

$\Rightarrow$ $\mathrm{Hom}(\underbrace{* \otimes N}_{**}, P)$ = l. exact $\forall P$

$\Rightarrow$ $**$ = r. exact. $\square$

Rmk: $T : \mathcal{A} \to \mathcal{B}$ $\quad U : \mathcal{B} \to \mathcal{A}$.

$$\mathrm{Hom}_{\mathcal{B}}(T(M), N) \cong \mathrm{Hom}_{\mathcal{A}}(M, U(N))$$

Then T is called the left adjoint of U

U . . . . . right . . . . . . T

Fact: T = right exact & U = left exact.

Pf: i) $* =$ r. exact$/\mathcal{A}$ $\Rightarrow$ $\mathrm{Hom}_{\mathcal{A}}(*, U(N)) =$ l. exact $\forall N$.

$\Rightarrow$ $\mathrm{Hom}_{\mathcal{B}}(T(*), N) =$ l. exact $\forall N$

$\Rightarrow$ $T(*) =$ r. exact$/\mathcal{B}$

ii) $\# =$ l. exact$/\mathcal{B}$ $\Rightarrow$ $\mathrm{Hom}_{\mathcal{B}}(T(M), \#) =$ l. exact

$\Rightarrow$ $\mathrm{Hom}_{\mathcal{A}}(M, U(\#)) =$ l. exact $\forall M$

$\Rightarrow$ $U(\#) =$ l. exact.

Rmk: $- \otimes N$ is not exact in general. e.g.

$A = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ $\Rightarrow$ $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}$.

An A-module N is called **flat**, if $-\otimes_A N$ is exact.

Prop 2.19 TFAE: ($N = A$-module)

i) $N =$ flat

ii) $\forall\ 0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow 0 \to M'\otimes N \to M \otimes N \to M''\otimes N \to 0$ exact

iii) $\forall\ M' \overset{f}{\hookrightarrow} M$ inj $\Rightarrow f\otimes 1 : M'\otimes N \to M \otimes N$ inj.

iv) $\left.\begin{array}{l}\forall\ M' \overset{f}{\hookrightarrow} M \text{ inj} \\ M', M = \text{f.g.}\end{array}\right\} \Rightarrow f\otimes 1 : M'\otimes N \to M\otimes N$ inj.

Pf: i) $\Leftrightarrow$ ii): long exact seq. splits into s.e.s.

ii) $\Leftrightarrow$ iii): 2.18

iii) $\Rightarrow$ iv): clear

iv) $\Rightarrow$ iii): $\forall\ \sum x_i' \otimes y_i \in \ker(M'\otimes N \to M \otimes N)$

$\Rightarrow \sum f(x_i') \otimes y_i = 0 \in M \otimes N$

$\Rightarrow \exists$ f.g. $M_0 < M$ s.t.

$f(x_i') \subset M_0$ & $\sum f(x_i')\otimes y_i = 0 \in M_0 \otimes N$

$M_0' := \sum_i A x_i' \subseteq M'$ f.g.

$$\Rightarrow \begin{array}{ccc} M_0'\otimes N & \hookrightarrow & M_0 \otimes N \\ \big\downarrow & \sum x_i'\otimes y_i \mapsto 0 & \big\downarrow \\ M'\otimes N & \xrightarrow{f\otimes 1} & M\otimes N \end{array} \Rightarrow \sum_i x_i' \otimes y_i = 0$$

Ex 2.20 : $f : A \to B$ ring. hom.

$$M/A = \text{flat} \Rightarrow M_B/B = \text{flat}.$$

Pf : $M_B \otimes_B * = M \otimes_A \underbrace{B \otimes_B *}_{= *}$

§2.10 Algebras

A-algebra = a ring B with ring hom. $f: A \to B$

= a ring equipped with an A-module structure

i.e. $\begin{array}{ccc} A & \longrightarrow & B \\ r & \longmapsto & r.1 \end{array}$ satisfy. $(r.1)(s.1) = rs.1$

Rmk: every ring is a $\mathbb{Z}$-alg.

<u>A-algebra homomorphism</u>

$$\begin{tikzcd} B \arrow[r, "h"] & C \\ & A \arrow[ul, "f"] \arrow[u, "g"'] \arrow[ul, phantom, shift left=3, "\circlearrowleft"] \end{tikzcd}$$

- $f: A \to B$ is finite (or, B = finite A-alg.)

  $\overset{\text{def}}{\Longleftrightarrow}$ B = f.g. as an A-module.

- $f: A \to B$ is finite type (or, B = finite type A-alg.)

  $\overset{\text{def}}{\Longleftrightarrow}$ $\exists\ A[t_1, \cdots, t_n] \twoheadrightarrow B$.

§2.11 Tensor product of algebras.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & B \otimes_A C =: D \end{tikzcd} \quad (\text{as an } A\text{-module})$$

Question: multiplication on D ?

$$B \times C \times B \times C \longrightarrow D$$
$$(b, c, b', c') \longmapsto (bb' \otimes cc')$$

$\Rightarrow \quad D \otimes D \to D \quad$ (A-mod. hom.)

$\Rightarrow \quad \mu : D \times D \to D \quad$ (A-bilinear)

$$(b \otimes c, b' \otimes c') \mapsto bb' \otimes cc'$$

(well-define !)

Fact : Together with μ, the D forms an A-alg.

Example : $A[x] \otimes_A A[y] \cong A[x, y]$

$$A/I \otimes_A A/J \cong A/(I+J)$$